From self-organized criticality to first-order-like behavior: A new type of percolative transition

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A modification of one of the standard models of avalanches in a sandpile is proposed. We allow *two* different values of the critical slope: a sliding never starts with a slope below the upper value and stops only when the local slope falls below the lower one. As a result we find that the *only* avalanches present involve the whole system, in sharp contrast with what is expected from self-organized critical models. This behavior resembles more a first-order transition than a second-order one. We present a method to study the gradual change from one behavior to the other. A possible link with percolative transitions is discussed.

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Self-organized criticality (SOC) is by now a well established framework for explaining the behavior of spatially extended dynamical systems [1–3]. The range of applicability of these ideas is very broad, including, among others, $1/f$ noise [1], earthquakes [4], magnetic-domain formation [5], pinned-flux lattices [6], Barkhausen effect [7], and some features of economic systems [8].

From the very first papers of Bak, Tang, and Wiesenfeld [1] they used a model of avalanches in sandpiles to convey the general ideas involved in SOC. In fact, almost all the literature published on this subject uses this as an archetypical example. This is most probably due to two different reasons: on the one hand, that image is very simple and easy to grasp; on the other hand, its implementation through cellular automata entails no special difficulties.

In a language appropriate to sandpiles, SOC predicts that when sand is slowly added to the pile, the system will dynamically adjust itself to a situation that, under perturbation, evolves by avalanches of all possible sizes, i.e., those having a variable number S of grains of sand that range from 1 to the size of the whole system. The distribution function $D(S)$ that measures the number of avalanches of size S (i.e., involving S grains) has been studied in great detail [3]. Among other features, it depends upon the dimensionality of the cellular automaton that is used to describe the pile and upon the updating rule. For the purposes of the present paper we will keep to the case studied by Bak, Tang, and Wiesenfeld [1], in which $D(S)$ follows a power law [i.e., $D(S) \sim S^{-\tau}$], characterized by a "critical index" $\tau \simeq 1$.

In order to check the consequences of the theory several experiments have been conducted on *real* sandpiles. The work of Held *et al.* [9] provided some support to the SOC model, particularly for small enough sandpiles. However, these experiments in larger piles also suggested that the occurrence of SOC is a finite-size effect. There is one recent experiment [10] in which it was found that the only avalanches present are roughly of the size of the whole system. Moreover, the experiment shows that the slope of the pile can slowly be increased and nothing happens until a "supercritical" slope θ_M is reached. A further increase gives rise to an avalanche that only stops when the slope falls below a minimum value $\theta_m < \theta_M$. These results have been described as being more similar to a first-order phase transition, rather than to critical phenomena.

These results indicate that further experiments are needed in order to know if sandpiles are a good example of systems displaying self-organized critical behavior. On the other hand, it is natural to consider the attractive problem of whether a single theoretical framework can be constructed for extended, open systems, capable of accounting for behaviors resembling those seen in second-order- *and* first-order-like transitions. The purpose of the present paper is to present such a framework.

Since the flow of sand in a pile is usually represented by cellular automata on a square lattice (of size $L \times L = N_T$), the above problem can be rephrased in a more concrete way as follows: what modifications have to be introduced in the standard cellular automata to find a behavior similar to a first-order phase transition? In what follows we agree to identify a "first-order-like" behavior, with a regime of avalanches characterized by the presence of hysteresis, a distinctive landmark of first-order transitions. In order to do so we proceed according to the following steps:

(i) We start as in Ref. [1(a)], preparing the system by (randomly) assigning to every automaton a local (integer) variable $z \gg z_c$. This represents the relative height of sand at the site with respect to its neighbors (the local slope). z_c is the threshold value for the slope of sand that in more general cases depends upon the number of dimensions of the lattice (the number of neighbors of each site). We will restrict our analysis to square lattices; therefore in what follows we take $z_c = 4$.

(ii) Next the lattice is allowed to relax to a stationary or inactive state $\mathcal{I}$ through "avalanches." In ordinary two-dimensional SOC models, a site with $z > z_c$ gives rise to an avalanche passing one grain of sand to each

of its four neighbors, thereby reducing its $z \rightarrow z - 4$. This process is assumed to take place in all sites of the lattice that are not on its border (the border acts as a sink: if a grain reaches it, is removed from the system). Within the present framework a hysteresislike property corresponds to increasing the slope beyond the critical value *without* inducing avalanches. This is very much the same as allowing for the possibility of overheating a liquid beyond the evaporating point. This calls for a change of this rule. We therefore assume that avalanches can *only* start when $z = z_c + 2$. In ordinary SOC models the state $\mathcal{I}$ involves only sites with $z \leq z_c$. With our modified rule, the new state $\mathcal{I}$ involves sites with $z \leq z_c + 1$. This change also requires a modification in the next step of the calculation, in which how avalanches propagate through the system becomes crucial.

(iii) After the state $\mathcal{I}$ is reached it is perturbed. Each perturbation gives rise to an avalanche whose size and lifetime are stored in order to construct point by point the functions $D(S)$ and $D(t)$. In ordinary SOC models an avalanche is induced by adding a single grain to a site in which $z = z_c$. In our framework, perturbations have to be made by adding a grain of sand to sites having a "supercritical" slope $z = z_c + 1$, since it is assumed that these are the only sites that can start avalanches. However, this is not enough. In this case a behavior presenting hysteresis is one in which avalanches start only if $z > z_c + 1$, but *once they have started* they continue to go as long as sites with $z \geq z_c$ are reached by a grain of sand. We therefore add the rule that if a site with $z = z_c$ is reached by a grain of sand from an avalanche that has started elsewhere, it should become active and propagate the flow of sand in a dominolike process: although sites with $z = z_c$ cannot *start* avalanches, they can *propagate* them if started elsewhere. By changing this and the preceding step we have assumed that there are two critical slopes: the avalanches start only when the slope is higher than the upper value and they stop only when it has dropped below the lower value. To construct the functions $D(S)$ and $D(t)$ all the "super-critical" sites of $\mathcal{I}$ are perturbed, one at a time, always starting from the same configuration.

(iv) To have more representative data, steps (i)–(iii) are repeated many times (we used 500 different initial states) and averages are taken. A coarse-graining average is also made because data are represented in a log scale. This "coarse graining" is performed by averaging data of a whole interval and assigning them to a "representative" point of it.

Having introduced these changes in the rules of individual cellular automata, we expect to find a change in the behavior of the system as a whole. A modification of the distribution of avalanches should be expected in such a way that the function $D(S)$ will be peaked near $S = N_T$.

We have performed numerical simulations on square lattices of size $L \times L$, with $L = 10$, 20, 30, 40, and 50. Averaging has been made over an ensemble of 500 samples. The effect of the changes is indeed overwhelming: essentially *all* avalanches are of the size of the whole lattice; i.e. to all practical purposes $D(S)$ is a δ function centered at $S = N_T$.

To understand this effect in a more detailed fashion we now turn to analyze how the system gradually approaches such a behavior. A controlled transition from self-organized criticality to first-order-like behavior can be induced in the following way:

(i′) Initially we allow the system to evolve into the inactive state with sites having $z \leq z_c$ as in ordinary SOC models. Relaxation therefore takes place through avalanches started at sites with $z = z_c$.

(ii′) Once this state is obtained, a small number of sites C are chosen at random and the corresponding local slopes are set to the value $z = z_c + 1$. We call this process "seeding." We obtain in this fashion a state $\mathcal{I}^*$ that resembles $\mathcal{I}$ in the sense that some of its sites have a "supercritical" slope. The main difference is that its density can be controlled in the seeding process.

(iii′) After seeding, we proceed as in step (iii) described above to construct the function $D(S)$. Each supercritical site is perturbed, one at a time, and always starting from the same state $\mathcal{I}^*$, and avalanches are induced. This

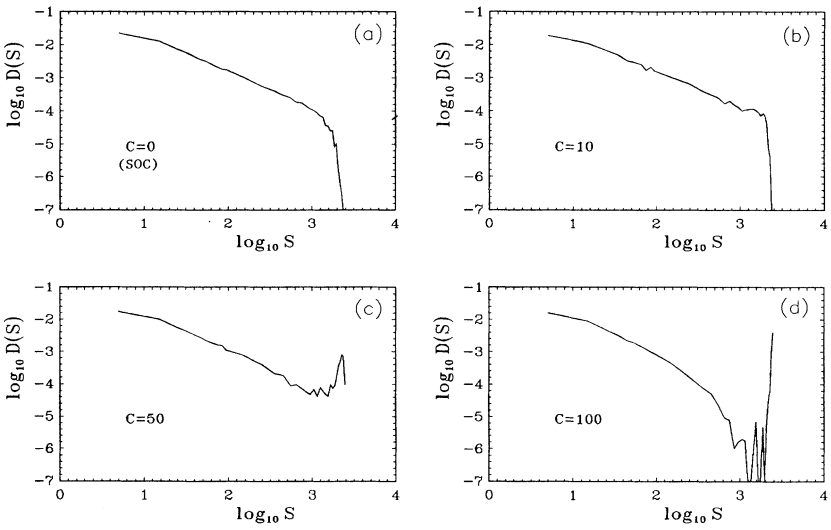

FIG. 1. Distribution function $D(S)$ as a function of S for a few representative values of C ($C = 10$, 50, 100) and a lattice of N_T =2500 sites. The curve for $C = 0$ corresponding to SOC is shown for comparison.

calculation can be done for any desired values of the density of implanted supercritical sites, therefore allowing a study of a whole variety of states $\mathcal{I}^*$ ranging from one almost identical to a SOC state to one with the same density of supercritical sites as $\mathcal{I}$.

(iv′) Averaging proceeds as before.

We have made numerical simulations for several lattice sizes (L = 10, 20, 30, 40, and 50), allowing a different number of seeds in state $\mathcal{I}^*$. We have considered C = 10, 20, 30, 50, 75, 100, 150, 200, and 250. In all cases the averaging mentioned in the step (iv′) has also been made over an ensemble of 500 samples.

The effect of an increasing density of supercritical sites can be observed in Fig. 1 in which we present the results for a lattice of 50 × 50 sites. As C increases one first observes that the dip produced in $D(S)$ due to the upper cutoff introduced by the finite size of the lattice is rapidly washed out. For bigger values of C, $D(S)$ develops a peak around $S \simeq N_T$. For even larger values of C the distribution becomes a δ-like function.

Before a full δ function has developed, the distribution $D(S)$ for low values of S can be approached by a power law. A most interesting result is obtained by plotting the slope τ of the log-log plot of that portion of $D(S)$, as a function of $x = C/N_T$. This has been done in Fig. 2 for the most representative lattice sizes we have considered. It is apparent that all points lie nearly on a straight line, thus indicating that τ has scaling, i.e., it is only a function of x and not of C and N_T separately; moreover, from our data we find that the exponent can be approximated by the straight line $\tau \simeq 5.54x + 1.01$.

We have made a preliminary study of the effect of the finite size of our samples. Figure 3 shows a finite-size scaling analysis of our data, while in Fig. 4 we represent the same data using a multifractal fit [3]. Notice that in both cases we have included size effects but not the effect of the number of seeds.

In Figs. 3 and 4 we show the distributions $D(S)$ for C = 50 and L = 30, 40 and 50, well below the value for

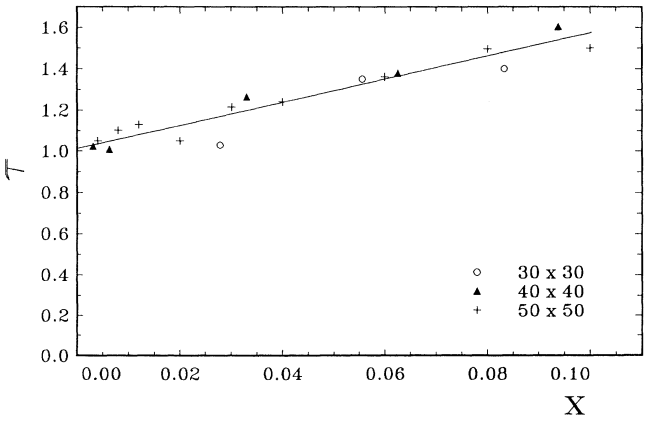

FIG. 2. The exponent τ vs $x = C/L^2$ for three lattice sizes considered in the simulations (C = 30, 40, and 50) and a different number of implanted seeds. The results of the numerical experiment have been fitted with the straight line $\tau = 5.54x + 1.01$.

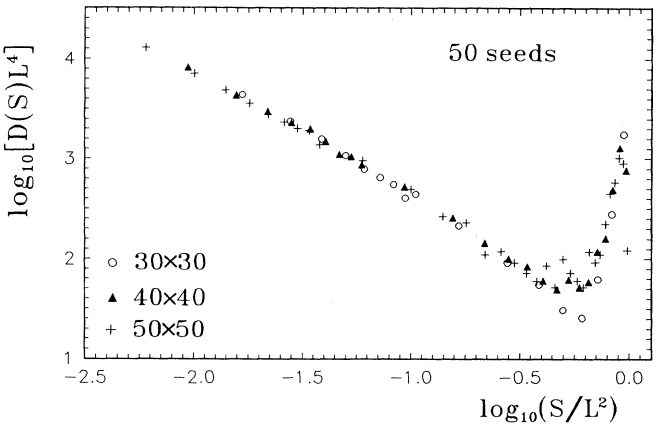

FIG. 3. Finite-size scaling analysis of the data for several lattice sizes.

which the δ-like distribution fully develops. Two regimes can easily be identified, one for very small and one for very large avalanches. At the small-S end a power law is adequate to fit $D(S)$, while at the other a very narrow peak develops. From Fig. 3 we conclude that our results are independent of the size of the sample. On the other hand, information about the crossover region between both regimes can be obtained from Fig. 4. The three rightmost small arrows indicate the beginning of these regions for every lattice size. Call $S^*(L)$ the corresponding avalanche size. To interpret these results [13] we calculate for L = 30, 40, and 50 the fraction $S^*(L)/L^2$ getting 0.65, 0.64, and 0.66, respectively. In other words, the departure from SOC starts to show up for avalanches covering $\approx 2/3$ of each sample, giving again a result independent of the lattice size. These values are only qualitative, because they are affected by sizable errors in the graphical extrapolations. More extensive calculations are needed in order to find the x dependence.

We believe that an attractive way of looking at the transition that occurs as the density C/N_T grows is to regard it as a *percolative transition* with clusters that involve all the sites of the lattice. A few years ago Tang

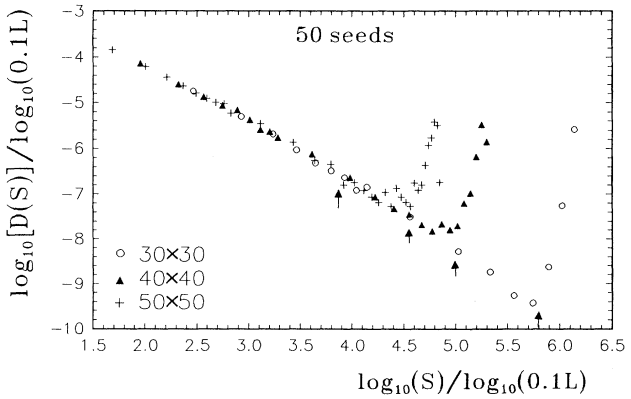

FIG. 4. Multifractal fit of the same data in Fig. 3. The arrows show the crossover regions.

and Bak [11] found that, within the mean-field approximation, the critical exponents of SOC and percolation were identical. They did not follow up this point, however, because numerical simulations indicated that these two phenomena were not in the same universality class. From a physical point of view it is clear, however, that the situation studied in this paper truly corresponds to a percolative transition: at a critical density of seeds, an avalanche that is started at any "supercritical" site propagates, reaching all sites of the lattice. There is no contradiction, however, because there is a clear difference between normal percolation [12] and the phenomenon studied here. In our case site probabilities are *dynamically changed* by the avalanches, and as a result percolation takes place before the critical value for the static percolation problem is reached. A very rough estimate of the new critical value is $x \leq 0.20$, which should be compared with 0.5 and 0.593, which correspond respectively to the bond and site critical values for the square lattice. The properties of this type of percolation require further study; work in this direction is in progress.

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[12] See, for instance, D. Stauffer, Phys. Rep. **54**, 1 (1979); or J. W. Essam, Rep. Prog. Phys. **43**, 833 (1980), and references therein.
[13] A plot of $\log_{10}S^*(L)/\log_{10}(0.1L)$ as a function of $1/L^2$ (including also the result for $L = 20$) shows that all these values fall in a *very good* straight line. A linear extrapolation for $1/L^2 \to 0$, however, cannot be done because it gives a nonsensical result (see the leftmost arrow in Fig. 4), namely, $S^* \sim L^{3.9}$, i.e., an avalanche greater than the system size.